Elementary Analysis 2060B 15/16 -Summary

1 differentiability

Definition 1.1. *Given $f : [a,b] \to \mathbb{R}$. We say that f is differentiable at $c \in (a,b)$ if there exists $L \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in (c-\delta, c+\delta) \setminus \{c\}$*

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \epsilon.$$

And we denote $f'(c) = L$.

Corollary 1.2. *If f is differentiable at c, then f is continuous at c.*

1.1 Properties of $f'(x)$

Theorem 1.3. *(Darbouxs Theorem, intermediate value property) If f is differentiable on $[a,b]$ and if $k \in (f'(a), f'(b))$, then there exist $c \in (a,b)$ such that $f'(c) = k$.*

Theorem 1.4. *(Mean value theorem) If f is differentiable on (a,b) and continuous on $[a,b]$, then there exists $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Here are some generalisations of mean value theorem.

Theorem 1.5. *(Cauchy Mean Value Theorem) Let f and g be continuous on $[a,b]$ and differentiable on (a,b), and assume that $g \neq 0$ for all x in (a, b). Then there exists $c \in (a,b)$ such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(a) - f(b)}{g(a) - g(b)}.$$

Consequence of Cauchy mean value theorem: L'Hpital's rule.

Theorem 1.6. *(Taylor's theorem) If f is k-th times differentiable on $[a,b]$, then there exists $c \in (a,b)$ such that*

$$f(b) = \sum_{n=0}^{k-1} \frac{f^n(a)}{n!}(b-a)^n + \frac{f^n(c)}{k!}(b-a)^k.$$

2 The Riemann Integral

We here follow the approach of Professor's Leung instead of that in the book.

Definition 2.1. *Given a bounded function f defined on $[a,b]$. We say that f is integrable if*

$$\inf\{U(f,\mathcal{P}) : \mathcal{P} \text{ is partition on } [a,b]\} = \sup\{L(f,\mathcal{P}) : \mathcal{P} : \mathcal{P} \text{ is partition on } [a,b]\} = L$$

where $U(f,\mathcal{P})$ is defined to be $\sum M_i \Delta x_i$, $M_i = \sup\{f(x) : x \in [x_i, x_{i+1}]\}$. Similar for $L(f,\mathcal{P})$. And we say $f \in R[a,b]$ and $\int_a^b f = L$

Proposition 2.1. *The followings are equivalent.*

1. $f \in R[a,b]$.

2. *For any* $\epsilon > 0$, *there exists partition* $\mathcal{P}$ *such that*

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon.$$

3. *For any* $\epsilon > 0$, *there exists* $\delta > 0$ *such that whenever* $\mathcal{P}$ *is a partition with* $||\mathcal{P}|| < \delta$, *then*

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon.$$

4. *There exists a unique* $A \in \mathbb{R}$ *such that for all* $\mathcal{P}$,

$$L(f,\mathcal{P}) \leq A \leq U(f,\mathcal{P}).$$

Remark: Noted that we will only concern bounded functions in the case of Riemann integrability.

Example 2.2. *Step functions, continuous functions, monotonic functions are integrable.*

Proposition 2.2. $R[a,b]$ *is a vector space over* $\mathbb{R}$. *That is to say addition, scalar multiplication are all valid.*

Once we know that $f \in R[a,b]$, we can find $\int_a^b f$ by any mean of approximation. Namely,

$$\sum_{i=1}^{n_k} f(t_j^k)\Delta x_i \to \int_a^b f$$

whenever $\mathcal{P}_k : a = x_0 < x_1 < ... < x_{n_k} = b$ and $t_i^k \in [x_i, x_{i+1}]$ with $||\mathcal{P}_k|| \to 0$.

2.1 Evaluation of integrals

Theorem 2.3. *(First form of fundamental theorem of Calculus) Suppose there is a finite set* E *on* $[a,b]$, $F : [a,b] \to \mathbb{R}$ *such that* F *is continuous,* $F' = f$ *for all* $x \in [a,b] \setminus E$ *and* $f \in R[a,b]$. *Then*

$$\int_a^b f = F(b) - F(a).$$

Corollary 2.4. *(Integration by parts) Let* F, G *be differentaible on* $[a,b]$. *If* F' *and* G' *are integrable, then*

$$\int_a^b F'(x)G(x)\,dx = F(x)G(x)\Big|_a^b - \int_a^b F(x)G'(x)\,dx.$$

2.2 Improper Integral

Definition 2.5. *Let $f : (a, b] \to \mathbb{R}$ be a real-valued function such that $f \in R[c, b]$ for all $c > a$. We say that the improper integral $\int_a^b f$ exists if there exists $L \in \mathbb{R}$ such that*

$$\lim_{c \to a} \int_c^b f = L.$$

Recall what we learnt from Math 2050. We have the following Cauchy type criterion.

Proposition 2.3. *The limit exists if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in (0, \delta)$, one have*

$$|\int_y^x f| < \epsilon.$$

Similar for the case when f is defined on a unbounded domain (see tutorial question list).

3 Sequence of functions

Definition 3.1. *Given a sequence of function f_n, $f : A \to \mathbb{R}$. We say that f_n converge to f pointwise if for each x, $f_n(x) \to f(x)$ as a sequence of real numbers. We say that f_n converge uniformly to f if*

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N}, \forall\, x \in A,\ \ |f_n(x) - f(x)| < \epsilon\ \forall n > N.$$

Proposition 3.1. *(cauchy criterion) f_n converges to f uniformly on A if and only if*

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall x \in A, |f_n(x) - f_m(x)| < \epsilon\ \forall m, n > N.$$

This statement is a bit stronger than that in the book. See Rudin's Principle of Mathematical Analysis for more details.

So, to show that the convergence is not uniform. It is equivalent to show the followings.

$$\exists \epsilon_0 > 0, x_k \in A,\ |f_{n_k}(x_k) - f_{m_k}(x_k)| \geq \epsilon_0 > 0.$$

3.1 Interchange of limiting process

Proposition 3.2. *Uniform convergence preserves continuity, integrability. More precisely, if $f_n \in C^0[a, b]$ ($R[a, b]$) and f_n converges uniformly to f on $[a, b]$, then $f \in C^0[a, b]$ ($R[a, b]$).*

Preservation of differentiability is a bit more restrictive. But we have

Proposition 3.3. *Suppose $f_n : [a, b] \to \mathbb{R}$ is a sequence of functions. If there exists $x_0 \in [a, b]$ such that $f_n(x_0) \to l \in \mathbb{R}$ and f_n' converges uniformly to a function g on $[a, b]$. Then f_n converges to some function $f : [a, b] \to \mathbb{R}$ with $f' = g$.*

The followings theorem is aimed to remove the assumption of uniform convergence in the integration theory.

Theorem 3.2. *(Bounded Convergence Theorem) Suppose $f_n \in R[a,b]$ and $f_n \to f \in R[a,b]$. If f_n is uniformly bounded on $[a,b]$, then*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Theorem 3.3. *(Dini's theorem) Suppose f_n is a monotonic sequence of continuous functions on $[a,b]$. If $f_n \to f \in C^0[a,b]$, then the convergence is uniform.*

4 Series of real numbers

We say that $\sum_{k=1}^{\infty} x_k$ is convergent if it is convergent as a partial sum.

Proposition 4.1. *(Divergence test) If $\sum_{k=1}^{\infty} x_k$ is convergent, then $x_n \to 0$.*

Similarly, we have cauchy criterion.

Proposition 4.2. *(Cauchy criterion) The series converges if and only if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$,*

$$\left| \sum_{k=m}^{n} x_k \right| < \epsilon.$$

To show the convergence, we mainly use the following test.

Proposition 4.3. *(comparsion test) Suppose a_n and b_n are sequcen of real numbers with $0 \le a_n \le b_n$. If $\sum b_n$ converges, then $\sum a_n$ converges.*

Theorem 4.1. *(monotone convergence theorem) If $a_n \ge 0$, then the series converges if and only if its partial sum is bounded above.*

Proposition 4.4. *(Root test) Denote $L = \limsup |a_n|^{1/n}$. Then if $L < 1$, the series is convergent absolutely. If $L > 1$, the series diverges.*

Proposition 4.5. *(Ratio test) Denote $L = \limsup |\frac{a_{n+1}}{a_n}|$. Then if $L < 1$, the series is convergent absolutely.*

Noted that in general, we have

$$\liminf |\frac{a_{n+1}}{a_n}| \le \liminf |a_n|^{1/n} \le \limsup |a_n|^{1/n} \le \limsup |\frac{a_{n+1}}{a_n}|.$$

Therefore, root test will obtain more informations than ratio test.

Proposition 4.6. *(Integral test) Let $f : [1, \infty) \to \mathbb{R}$ be a continuous function which is non-negative and decreasing. Then the series $\sum f(n)$ exists if and only if the improper integral $\int_1^{\infty} f$ exists.*

Example 4.2. *The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ exists if $p > 1$.*